Unified model for partially coherent solitons in logarithmically nonlinear media

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We investigate the propagation of a partially coherent beam in a nonlinear medium with logarithmic nonlinearity. We show that all information about the properties of the beam, as well as the condition for formation of incoherent solitons, can be obtained from the evolution equation for the mutual coherence function. The key parameter is the detuning Δ between the effective diffraction radius and the strength of the nonlinearity. Stationary partially coherent solitons exist when $\Delta=0$ and the nonlinearity exactly compensates for the spreading due to both diffraction and incoherence. For nonzero detunings the solitons are oscillating in nature, and we find approximate solutions in terms of elliptic functions. Our results establish an elegant equivalence among several different approaches to partially coherent beams in nonlinear media.

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Among recent advances in the field of optical solitons, demonstration of the formation of incoherent (or partially coherent) spatial solitons [1–12] has attracted particularly strong attention as it opens the possibility of using light sources with degraded or poor coherence in soliton-based all-optical signal processing. Typically, spatial solitons are created by self-trapping coherent optical beams, i.e., beams whose phase at any two points is fully correlated. Such beams differ significantly from those generated by an incoherent light source in which there is no correlation between light emitted from two different points. This results in some level of randomness (or partial correlation) in the phase across the beam. The weaker the phase correlation, the stronger the incoherence [13]. As a result, a partially coherent beam spreads faster than its coherent counterpart of the same width. Additionally, the intensity distribution across the beam exhibits a speckle structure, which prevents the ''standard'' uniform self-focusing observed in instantenous nonlinear media as the beam tends to form filaments.

It turns out, however, that self-focusing and soliton formation are still possible provided the nonlinear medium is inertial and responds much slower than the time scale characterizing the random phase variation. In such cases the medium will respond to the time averaged intensity, which, being a smooth function of the spatial variables, will induce a smooth waveguidelike structure trapping the beam. In fact, the first experiments with incoherent solitons were conducted using the relatively slow photorefractive nonlinearity [1,2,6].

The self-focusing of partially coherent light in inertial (slow) nonlinear Kerr-like materials was originally suggested by Akhmanov *et al.* in 1967 [14], and studied subsequently by Pasmanik [15] and Aleshkevich *et al.* [16]. On the other hand, the formation of temporally incoherent solitons was suggested and investigated by Hasegawa [17].

There are several approaches as far as the theoretical description of propagation of partially coherent beams in a slow nonlinear medium is concerned. In recent works, the coherent density approach has been sucessfully applied to find incoherent soliton solutions with stationary intensity profiles in saturable media. This approach is based on representing the beam as a superposition of mutually incoherent components [3]. For the case of a logarithmic nonlinearity the exact analytical stationary soliton solution has been found [4]. In another approach, the stationary soliton solution for a logarithmic nonlinearity was found using a multimode decomposition of the field [9]. Finally, in the diffractionless limit, the geometric optics approach was also used [11,12]. However, this approach is only valid when the size of the beam is large compared to the wavelength. In this limit, diffraction can be neglected, and the spreading of the beam is determined solely by its incoherence.

The most natural way of treating the propagation of a partially coherent beam is to use the so-called mutual coherence function [13], which gives a measure of the correlation between the amplitude of the field at two different points in the beam. Unfortunately, while rigorous, this method often leads to analytically unsolvable nonlinear equations which have to be dealt with either by approximate methods [15] or numerically [16]. However, in the special case of logarithmic nonlinearity this rigorous approach leads to a closed form analytical solution for the evolution of stationary partially coherent beams.

In this paper we show that the evolution equation for the coherence function in a logarithmically nonlinear medium has an exact analytical stationary solution for partially coherent beams. We show that this special nonlinearity allows a rigorous use of the Gaussian-Schell model for the optical beam [13], with its Gaussian statistical properties being maintained throughout propagation in the nonlinear medium.

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This is not the case in, for example, Kerr media [16,20], where nonlinearity actually induces changes in the beam's statistical properties. We use these solutions to find conditions for the formation of stationary solitons, and compare them with those found using completely different approaches. We go one step further, and show that the stationary solitons are only a special case of a much larger class of periodic solitons, for which we find approximate analytical solutions. Finally, we show that this method can be successfully applied to treat the evolution of elliptical partially coherent beams.

Let us consider the propagation of a partially coherent beam in a slow nonlinear bulk medium which responds to the time averaged intensity. We start with a paraxial wave equation describing propagation of a two-dimensional quasi-monochromatic partially coherent beam with the amplitude $\psi(\vec{r})$:

$$i\frac{\partial\psi}{\partial z}+\frac{1}{2}\nabla_{\vec{r}}^{2}\psi+\delta n(I)\psi=0, \tag{1}$$

where $\vec{r}=(x,y)$. The z dependence is understood, unless specifically given. The properties of the partially coherent beam are best described by the mutual coherence function $\Gamma(\vec{r}_1,\vec{r}_2)$, defined as

$$\Gamma(\vec{r}_1,\vec{r}_2)=\langle\psi(\vec{r}_1)\psi^*(\vec{r}_2)\rangle, \tag{2}$$

where brackets denote temporal or ensemble averaging. In particular, the time averaged intensity is obtained from the coherence function as $I(\vec{r})=\Gamma(\vec{r},\vec{r})$. We take the refractive index change δn to be a logarithmic function of the intensity:

$$\delta n(I)=n_2\ln I. \tag{3}$$

It is straightforward to show that the mutual coherence function satisfies the differential equation (see also Refs. [12,15,16])

$$i\frac{\partial\Gamma_{12}}{\partial z}+\frac{1}{2}(\nabla_{\vec{r}_1}^{2}-\nabla_{\vec{r}_2}^{2})\Gamma_{12}+[\delta n(\vec{r}_1)-\delta n(\vec{r}_2)]\Gamma_{12}=0, \tag{4}$$

where we have defined $\Gamma_{ij}=\Gamma(\vec{r}_i,\vec{r}_j)$, $i,j=(1,2)$. Introducing the independent spatial variables $\vec{R}$ and $\vec{p}$,

$$\vec{R}=\frac{1}{2}(\vec{r}_1+\vec{r}_2),\quad \vec{p}=\vec{r}_1-\vec{r}_2, \tag{5}$$

and using the specific form [Eq. (3)] of the nonlinearity, transforms the propagation equation into

$$i\frac{\partial\Gamma_{12}}{\partial z}+\vec{\nabla}_{\vec{R}}\cdot\vec{\nabla}_{\vec{p}}\Gamma_{12}+n_2\ln\left(\frac{\Gamma_{11}}{\Gamma_{22}}\right)\Gamma_{12}=0. \tag{6}$$

In order to find solutions to this equation, we assume that the incident beam $\psi(\vec{r})$ possesses Gaussian statistics which implies that the coherence function initially has the form

$$\Gamma(\vec{r}_1,\vec{r}_2,z=0)=\exp\left(-\frac{r_1^2+r_2^2}{2\rho_0^2}-\frac{|\vec{r}_1-\vec{r}_2|^2}{r_c^2}\right), \tag{7}$$

where ρ_0 and r_c denote the initial diameter and coherence radius of the beam, respectively. In the variables $\vec{R}$ and $\vec{p}$ this function becomes

$$\Gamma(\vec{R},\vec{p},z=0)=\exp\left(-\frac{R^2}{\rho_0^2}-\frac{p^2}{\sigma_0^2}\right), \tag{8}$$

where $R=|\vec{R}|$, $p=|\vec{p}|$, and we have introduced the effective coherence radius $1/\sigma_0^2=1/r_c^2+1/(4\rho_0^2)$. Due to the logarithmic form of the nonlinearity, $\psi(\vec{r})$ will maintain the Gaussian statistics during propagation, and thus the coherence function will keep the form of Eq. (8). Furthermore, the input amplitude plays no role and is set to unity. We can therefore look for solutions to Eq. (6) using the Gaussian anzatz

$$\Gamma(\vec{R},\vec{p},z)=A(z)\exp\left(-\frac{R^2}{\rho^2(z)}-\frac{p^2}{\sigma^2(z)}+i\vec{R}\cdot\vec{p}\mu(z)\right), \tag{9}$$

where $A(z)$ and $\mu(z)$ represent the amplitude and phase variation of the coherence function, and $\rho(z)$ and $\sigma(z)$ its diameter and coherence radius, respectively. The initial conditions are $A(0)=1$, $\rho(0)=\rho_0$, $\sigma(0)=\sigma_0$, and $\mu(0)=0$. Inserting these expressions into Eq. (6), we obtain a set of ordinary differential equations for the parameters of the coherence function:

$$\frac{d\sigma}{dz}=\sigma\mu, \tag{10}$$

$$\frac{d\rho}{dz}=\rho\mu, \tag{11}$$

$$\frac{dA}{dz}=-2A\mu, \tag{12}$$

$$\frac{d\mu}{dz}=\frac{4}{\sigma^2\rho^2}-\mu^2-\frac{2n_2}{\rho^2}. \tag{13}$$

From the first two equations we obtain the relation

$$\sigma/\rho=\sigma_0/\rho_0, \tag{14}$$

which shows that during evolution the beam conserves its coherence, defined as the number of speckles within the beam diameter. The wider (narrower) the beam, the larger (smaller) the coherence radius. Combining Eqs. (11) and (12) gives the amplitude $A(z)=[\rho_0/\rho(z)]^2$. Finally, inserting Eq. (13) into Eq. (11), we obtain the evolution equation

$$\frac{d^2\rho}{dz^2}-\frac{4}{\rho^3}\frac{\rho_0^2}{\sigma_0^2}+\frac{2n_2}{\rho}=0, \tag{15}$$

describing the dynamics of the width $\rho(z)$ of a partially coherent beam (with Gaussian statistics) in a logarithmic nonlinear medium. Clearly, the dynamics is determined by a competition between free spreading and nonlinearity. If $n_2=0$, then Eq. (15) describes simple diffraction of the partially coherent beam. A defocusing nonlinearity with $n_2<0$

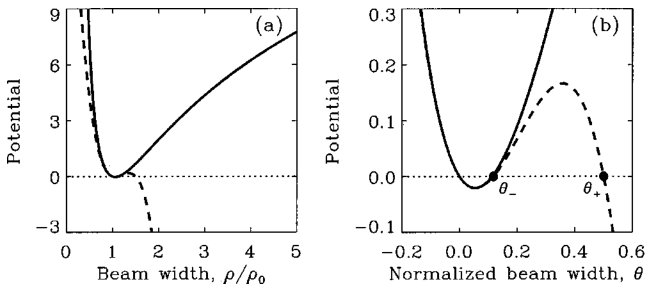

FIG. 1. Potential $P(\rho)$ (solid) and its third order Taylor expansion $P_3(\rho)$ (dashed) for an initial beam width of $\sigma_0=1$ and a focusing logarithmic nonlinear medium with $n_2=1.8$.

simply enhances the natural spreading of the beam, and we will not consider it here. Instead we will concentrate on the focusing case with $n_2>0$.

Choosing the initial condition such that $(d\rho/dz)(z=0)=0$, and integrating Eq. (15) once, we find that the evolution of $\rho(z)$ is described by Newton's equation for an effective particle,

$$(d\rho/dz)^2+P(\rho)=0, \tag{16}$$

moving in the potential $P(\rho)$, which is given by

$$P(\rho)=\frac{4}{\sigma_0^2}\left(\frac{\rho_0^2}{\rho^2}-1\right)+4n_2\ln\left(\frac{\rho}{\rho_0}\right). \tag{17}$$

The asymmetric potential is depicted in Fig. 1 for a beam of diameter $\sigma_0=1$ moving in a focusing logarithmic nonlinear medium with $n_2=1.8$.

Stationary soliton solutions with a constant beam width (corresponding to the effective particle being located at the bottom of the potential well) are formed at zero detuning:

$$\Delta\equiv n_2-\frac{2}{\sigma_0^2}=n_2-\frac{2}{r_c^2}-\frac{1}{2\rho_0^2}=0. \tag{18}$$

Physically, this condition means that soliton existence requires the nonlinearity-induced focusing to compensate for beam spreading due to both diffraction and incoherence. The diameter of the stationary partially coherent soliton is given by

$$\rho_0^2=\frac{1}{2n_2-4/r_c^2}. \tag{19}$$

We note that for perfectly coherent beams, i.e., for $r_c=\infty$, we recover the known solution for coherent solitons in logarithmically nonlinear media [18,19]. On the other hand, Eq. (19) also shows that a soliton cannot exist if the coherence radius of the input beam is lower than $\sqrt{2/n_2}$. Since the inverse of the coherence radius corresponds to the width of the incoherent spectrum of the partially coherent beam, this reproduces the result earlier obtained by Christodoulides *et al.* using both a coherent density approach [4] and modal decomposition [9]—an indication that all of these theoretical descriptions of the beam are equivalent.

At this point it is worth noting that the geometric optics approach to partially coherent solitons used in Refs. [11,12],

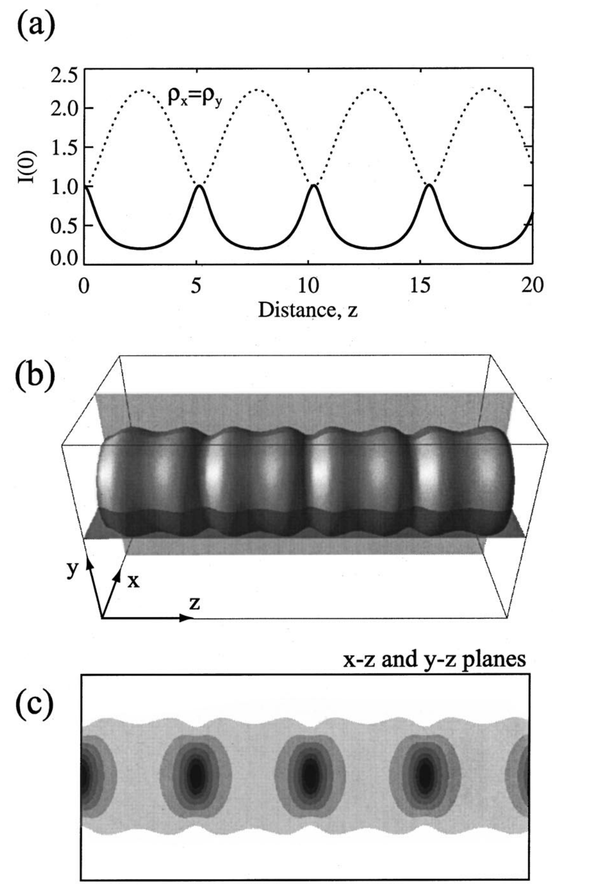

FIG. 2. Nonstationary propagation of a cylindrical partially coherent beam in a logarithmic nonlinear medium. (a) Beam diameter (dotted) and peak intensity (solid) as functions of the propagation distance. (b) Three-dimensional view of the beam with an intensity isosurface at 10% of the peak value. (c) Longitudinal cross section of the beam.

gives (in the case of logarithmic nonlinearity) a soliton solution of arbitrary diameter. However, this approach is restricted to cases in which diffraction is negligible, and spreading of the beam is caused solely by incoherence (diffusive illumination).

For nonzero detuning $\Delta\neq 0$, the beam diameter (as well as the coherence radius) will undergo periodic oscillations, corresponding to the effective particle oscillating in the bottom of the potential well. These general oscillating solutions were recently discussed in Ref. [12], where incoherent solitons were studied using a geometric optics approximation.

We can find an approximate analytical expression for these oscillating solutions when the detuning is small. In this case $\rho(z)$ will remain close to the initial value ρ_0, and we can expand the potential around $\rho=\rho_0$. To third order $P(\rho)\approx P_3(\rho)$, where $P_3(\rho)$ is given by

$$P_3(\rho)=4(\Delta\theta+\alpha_1\theta^2+\alpha_2\theta^3), \quad \theta=\frac{\rho}{\rho_0}-1. \tag{20}$$

Here $\alpha_1=3/\sigma_0^2-n_2/2$ and $\alpha_2=n_2/3-4/\sigma_0^2$. Using potential (20) in Eq. (16), we can integrate it by quadrature and obtain the solution in terms of the Jacobi elliptic sn function [21]

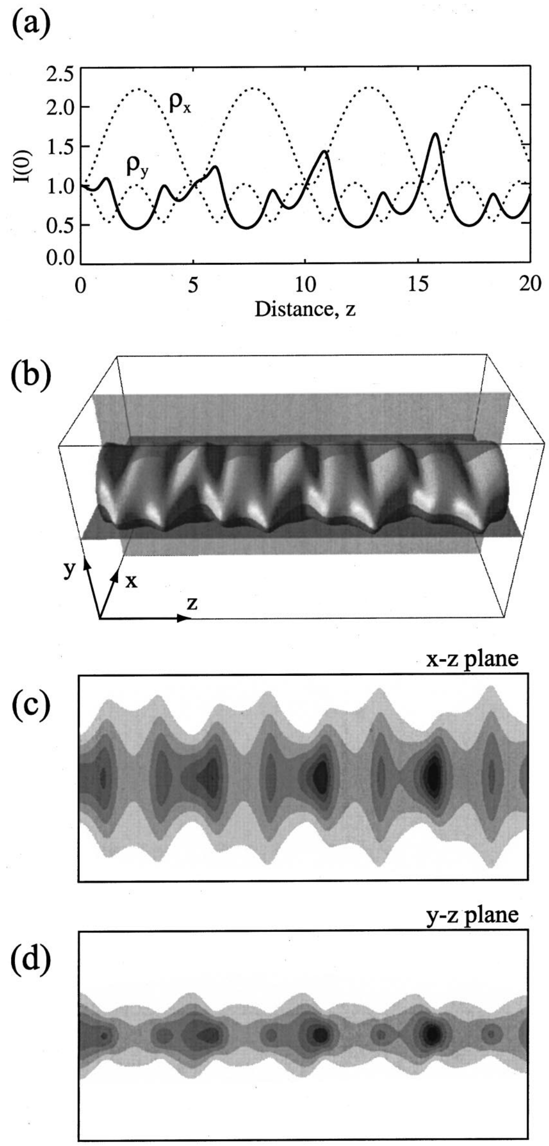

FIG. 3. Nonstationary propagation of an elliptical partially coherent beam in a logarithmic nonlinear medium. (a) Beam radii (dotted) and peak intensity (solid) as a function of propagation distance. (b) Three-dimensional view of the beam. (c) and (d) Longitudinal cross sections of the beam.

$$\rho(z)=\rho_0\left[1+\theta_- \mathrm{sn}^2\left(\frac{\sqrt{\theta_+\alpha_2}}{\rho_0}z;m\right)\right], \quad m=\frac{\theta_-}{\theta_+}. \quad (21)$$

Here $\theta_\pm$ are the solutions of the quadratic equation $\alpha_2\theta^2+\alpha_1\theta+\Delta=0$. Using $|\Delta|\ll 1$, we obtain $\theta_-\approx-\Delta/n_2$ and $\theta_+\approx 3/5$.

Thus the beam width will oscillate between ρ_0 and another value, where the potential is negative. For positive detuning the nonlinear self-focusing dominates, and the beam width decreases initially, i.e., it oscillates between ρ_0 and a somewhat lower value. For negative detuning the diffraction dominates and the beam width increases initially, i.e., it oscillates between ρ_0 and a somewhat larger value.

The analysis presented above can easily be extended to the case of elliptical beams. Let us denote the initial beam diameter along the x and y axes by ρ_{x0} and ρ_{y0}, respectively, and the corresponding initial coherence radii by r_{cx} and r_{cy}. Then it can be shown that the dynamics of the beam radii can be described by the following set of differential equations:

$$\frac{d^2\rho_x}{dz^2}-\frac{4}{\rho_x^3}\frac{\rho_{x0}^2}{\sigma_{x0}^2}+\frac{2n_2}{\rho_x}=0, \quad (22)$$

$$\frac{d^2\rho_y}{dz^2}-\frac{4}{\rho_y^3}\frac{\rho_{y0}}{\sigma_{y0}^2}+\frac{2n_2}{\rho_y}=0,$$

where

$$\frac{1}{\sigma_{x0}^2}=\frac{1}{r_{cx}^2}+\frac{1}{4\rho_{x0}^2}, \quad \frac{1}{\sigma_{y0}^2}=\frac{1}{r_{cy}^2}+\frac{1}{4\rho_{y0}^2}. \quad (23)$$

It is evident from Eq. (22) that the dynamics along both principal axes are completely uncoupled. In general, a partially coherent Gaussian beam of elliptical shape propagating in a logarithmic nonlinear medium will experience periodic oscillations along both axes. Further, an elliptically shaped stationary soliton can be formed if the coherence parameters of the beam and its diameters are given by the relations

$$\rho_{x0}^2=\frac{1}{2n_2-4/r_{cx}^2},$$

$$\rho_{y0}^2=\frac{1}{2n_2-4/r_{cy}^2}. \quad (24)$$

Again, these are exactly the same conditions as those obtained previously using the modal decomposition of the self-induced optical waveguide [9]. Interestingly, in the geometric optics limit, although the shape of the soliton can be arbitrary, the soliton coherence function must be isotropic. This is a direct consequence of the fact that in this limit the beam spreading due to diffraction is neglected. Taking $\rho_{x0}=\rho_{y0}=\infty$ in Eq. (24), we obtain the soliton condition

$$r_{cx}=r_{cy}=2/n_2, \quad (25)$$

i.e., nonlinearity compensates for the spreading of the beam induced solely by its incoherence. It should also be noted that in the case of an elliptical beam its dynamics in the vicinity of the stationary solution can be described by approximate analytical expressions. For small detunings ($\Delta_x\equiv n_2-2/\sigma_{x0}^2$ and $\Delta_y\equiv n_2-2/\sigma_{y0}^2$), the oscillations of the beam diameters are again given in terms of the Jacobi elliptic functions.

To illustrate our results, in Figs. 2 and 3 we show the nonstationary propagation of two partially coherent beams. In Fig. 2, the initial parameters are chosen such that $\rho_{x0}=\rho_{y0}=1.0$, $r_{cx}=r_{cy}=1.15$, and $n_2=1$. In the top graph we plot the beam radii (dotted lines), as well as the peak intensity of the beam (solid line) as functions of the propagation distance. To emphasize the three-dimensional nature of the beam, Fig. 2(b) shows an isosurface of the beam intensity (thresholded at 10% of the peak value) along with two orthogonal cut planes whose intensity is displayed in Fig. 2(c). In this particular case, the circularly symmetric beam exhibits periodic contractions and expansions during propagation.

Since the detuning is negative, the diffraction initially prevails and evolution of the beam starts with expansion.

Figure 3 shows the nonstationary propagation of an elliptical partially coherent beam. Here the initial conditions are the same as in Fig. 2, with the exception of the y axis coherence radius whose value is changed to $r_{cy}=2.3$. As detunings along principal axes now have different signs ($\Delta_x<0,\Delta_y>0$) the beam diameter along the y axis initially decreases, while that along the x axis increases. For numerical values of the initial parameters used in this simulation the principal beam radii oscillate with incommensurate periods, and the peak intensity exhibits quasiperiodic oscillations. The complexity of the overall intensity distribution is shown in Figs. 3(b)–3(d), where now the two orthogonal cut planes display differing intensity patterns.

In conclusion, we have presented a rigorous analysis of the propagation of partially coherent beams with Gaussian statistics in a logarithmically nonlinear medium. Our approach is based on the evolution of the mutual coherence function, and is able to capture the dynamics of the beam diameter as well as its coherence properties simultaneously. This results in relatively simple equations governing the dynamics of important beam parameters. It appears that this dynamics is determined by the detuning Δ between the effective diffraction radius and the strength of the nonlinearity. We showed that this leads to the same conditions for stationary soliton formation as those obtained previously using coherent density and multimode decomposition methods, thereby indicating their equivalence.

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